

# Quadratic Solitons in Negative Refractive Index Medium

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## ABSTRACT

We are considered the propagation of the fundamental and second harmonic solitary waves under the slowly varying envelope pulses approximation in a quadratic nonlinear medium that characterized by negative refraction index at the frequency of fundamental wave and by positive refractive index at the second harmonic frequency. We find the some solutions of the evolution equations which described the steady state simulton, coupled second harmonic and fundamental frequency waves propagation in this materials.

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## 1 Introduction

In the past few years, new developments in structured electromagnetic materials have given rise to negative refractive index materials which have both negative dielectric permittivity and negative magnetic permeability in some frequency ranges. These materials are often referred to as *left-handed materials* (LHM) or materials with negative refraction (NRIM). The properties of such materials were analyzed theoretically by Mandelshtam and Veselago

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many years ago [1, 2]. It is very important that the wave vector is antiparallel with the Poynting vector inside LHM [1]-[7]. The occurrences of the LHM in microwave region have been demonstrated experimentally in the [8]-[11]. The experimental evidence for the existence of NRIM in optical region was found [12]-[16] also. The results of investigation and discussion of the applications of negative refractive medium represent in reviews [17, 18].

Unusual features of the NRIM manifest itself in the passing the interface between this material and the positive refractive index one. Also, the refractive index of the same medium can be positive in one frequency region and to be negative in another frequency region. Hence the features of NRIM would be made itself evident in interaction of wave packets having carrier frequencies from different spectral regions, where refractive indexes have opposite signs [23, 24]. Second harmonic generation (SHG) is first example where negative refraction leads to distinctly different pictures of spatial distributions of the interacting waves intensity [25, 26]. Here the quadratic nonlinear NRIM acts similar to distributed Bragg reflector, i.e., Bragg gratings. Namely, the harmonic wave propagates towards fundamental (pump) wave. In nonlinear Bragg grating under some conditions the connected pair of the incident and reflected waves develops. It is gap soliton propagating in nonlinear grating [27]. We can expect that in quadratic nonlinear NRIM the second harmonic and fundamental wave pulses are able to produce the complex steady state solitary wave that is similar to quadratic soliton [21, 22] or simulton [28].

In this paper the propagation of the complex steady state of the fundamental wave and second harmonic wave considered. It is assumed that refractive index for the frequency of fundamental wave is negative, and refractive index at the harmonic wave frequency is positive. The particular solutions of the equations describing the evolution of these waves are founded. They are analogues of the solitons [21, 22] and cnoidal waves [29, 30] in quadratic nonlinear medium. Under condition that value of the phase mismatch is very large the propagation interacting waves approximately governs by nonlinear Schrödinger equation [31, 32].

## 2 Constituent equations

Consider the propagation of the pulses of parametrically coupled waves, i.e., fundamental wave (or pump wave) and second harmonic one. The type I phase-matching condition is taking place [19, 20]. In this case the slowly varying envelopes of pump and harmonic waves evolve according to following system of equations can be written in following form

$$\begin{aligned} i \left( \hat{k}_1 \frac{\partial}{\partial z} + \frac{1}{v_1} \frac{\partial}{\partial t} \right) \tilde{E}_1 - \frac{D_1}{2} \frac{\partial^2}{\partial t^2} \tilde{E}_1 &= -\frac{2\pi\omega_1^2\mu(\omega_1)}{c^2k_1} \tilde{P}_{NL}(\omega_1) \exp(-ik_1z) (1) \\ i \left( \hat{k}_2 \frac{\partial}{\partial z} + \frac{1}{v_2} \frac{\partial}{\partial t} \right) \tilde{E}_2 - \frac{D_2}{2} \frac{\partial^2}{\partial t^2} \tilde{E}_2 &= -\frac{2\pi\omega_2^2\mu(\omega_2)}{c^2k_2} \tilde{P}_{NL}(\omega_2) \exp(-ik_2z) (2) \end{aligned}$$

where  $\omega_1 = \omega_p, \omega_2 = \omega_s = 2\omega_p, k_j^2 = (\omega_j/c)^2 \varepsilon(\omega_j) \mu(\omega_j)$ ,  $\hat{k}_j$  is sign of square root of  $k_j^2$ , and

$$\begin{aligned} \tilde{P}_{NL}(\omega_1) &= \chi_2(\omega_1; \omega_2, -\omega_1) \tilde{E}_2 \tilde{E}_1^* \exp[iz(k_2 - k_1)], \\ \tilde{P}_{NL}(\omega_2) &= \chi_2(\omega_2; \omega_1, \omega_1) \tilde{E}_1^2 \exp[iz(k_1 + k_1)]. \end{aligned}$$

where  $D_{1,2}$  are dispersion coefficients at pump and signal wave frequencies [21,

22]. Here  $\tilde{E}_1$  is slowly varying envelope of the pump (fundamental frequency) wave, and  $\tilde{E}_2$  is the slowly varying envelope of the second harmonic wave.

As we have in equations (1,2) ratio  $(\omega_j/c) \sqrt{\mu(\omega_j)/\varepsilon(\omega_j)}$ , the coefficients in right part of equations (1,2) are not depending on signs of the dielectric permittivity and magnetic permeability. Nevertheless direction of the wave propagation is defined by signs  $\hat{k}_j$ .

It should be pointed that we consider the collinear propagation and so the collinear phase mismatch. The condition  $\Delta k = 0$  means that vector  $\vec{k}_2$  of pump wave is collinear and equal to vector  $2\vec{k}_1$ . The configuration of these vectors defines the energy flow directions. If the nonlinear medium is characterized by positive refraction index, then vectors  $\vec{k}_{1,2}$  and Pointing vectors have same directions. Opposite figure take place if the nonlinear medium is characterized by negative refraction index. In this case waves with frequencies from frequency region of negative refraction index are directed according to zero phase mismatch condition, but the Pointing vectors are contra-directed [23, 25] (see Fig.1).

We will consider the SHG under assumption that frequency of pump wave belong the frequency region of negative refraction index, while the frequency of the signal wave lies in frequency region positive refraction index.

Let us introduce the normalized variables

$$\zeta = z/L, \tau = (t + z/v_1)/t_p, \quad \tilde{E}_1 = A_{10}e_1, \tilde{E}_2 = A_{20}e_2 \exp(iz\Delta k), \quad (3)$$

$$A_{10} = c(2\pi\omega_1 L \chi_2(\omega_1))^{-1} \sqrt{\varepsilon(\omega_1)/\mu(\omega_1)}, \quad (4)$$

$$A_{20} = c(2\pi\omega_2 L \chi_2(\omega_2))^{-1} \sqrt{\varepsilon(\omega_2)/\mu(\omega_2)}, \quad (5)$$

$$\chi_1 = \chi^{(2)}(\omega_1; 2\omega_1, -\omega_1), \quad \chi_2 = \chi^{(2)}(2\omega_1; \omega_1, \omega_1), \quad (6)$$

$$\mu_{1,3} = \mu(\omega_{1,3}), \quad \varepsilon_{1,3} = \varepsilon(\omega_{1,3}), \quad (7)$$

where  $\Delta k = k_2 - 2k_1$ . Then SHG is described by the following normalized system of the equations

$$i \frac{\partial}{\partial \zeta} e_1 + \frac{\sigma}{2} \frac{\partial^2}{\partial \tau^2} e_1 - e_1^* e_2 = 0 \quad (8)$$

$$i \left( \frac{\partial}{\partial \zeta} + \delta \frac{\partial}{\partial \tau} \right) e_2 - \frac{\beta}{2} \frac{\partial^2}{\partial \tau^2} e_2 - \Delta e_2 + \frac{\theta}{2} e_1^2 = 0 \quad (9)$$

where

$$\delta = L t_p^{-1} (v_1^{-1} + v_2^{-1}) > 0, \quad \theta = \left( \frac{2\chi_1^{(2)}}{\chi_2^{(2)}} \right)^2 \sqrt{\frac{\mu(\omega_1)\varepsilon(2\omega_1)}{\mu(2\omega_1)\varepsilon(\omega_1)}} \approx 1$$

$$\sigma = \text{sgn} D_1, \quad \beta = D_2/|D_1|, \quad L = t_p^2/|D_1|.$$

Parameter  $\delta$  takes account of the walk-off effect for pump and harmonic pulses that is due to the difference of the group velocities directions for interaction waves. This parameter can not be zero, as in the case of positive refractive index medium.

### 3 Integral of motion - the Manley-Rowe relation

By using the equations (8,9) the following expression can be obtained

$$i \frac{\partial}{\partial \zeta} |e_1|^2 + \frac{\sigma}{2} \left( e_1^* \frac{\partial^2 e_1}{\partial \tau^2} - e_1 \frac{\partial^2 e_1^*}{\partial \tau^2} \right) - e_1^{*2} e_2 + e_2^* e_1^2 = 0,$$

$$i\left(\frac{\partial}{\partial \zeta} + \delta \frac{\partial}{\partial \tau}\right)|e_2|^2 - \frac{\beta}{2} \left( e_2^* \frac{\partial^2 e_2}{\partial \tau^2} - e_2 \frac{\partial^2 e_2^*}{\partial \tau^2} \right) - \frac{1}{2} (e_1^{*2} e_2 - e_2^* e_1^2) = 0.$$

From these equations it follows that

$$\begin{aligned} 0 = & i \frac{\partial}{\partial \zeta} \left( |e_2|^2 - \frac{1}{2} |e_1|^2 \right) + i \delta \frac{\partial}{\partial \tau} |e_2|^2 - \\ & - \frac{\beta}{2} \frac{\partial}{\partial \tau} \left( e_2^* \frac{\partial e_2}{\partial \tau} - e_2 \frac{\partial e_2^*}{\partial \tau} \right) - \frac{\sigma}{4} \frac{\partial}{\partial \tau} \left( e_1^* \frac{\partial e_1}{\partial \tau} - e_1 \frac{\partial e_1^*}{\partial \tau} \right). \end{aligned}$$

If we take into account the boundary condition

$$|e_{1,2}|^2 \rightarrow |e_{10,20}|^2 = \text{const},$$

or

$$|e_{1,2}|^2 \rightarrow 0$$

at  $|\tau| \rightarrow \pm\infty$ , then

$$\int_{-\infty}^{\infty} \left( |e_2|^2 - \frac{1}{2} |e_1|^2 \right) d\tau = \text{const}.$$

In the case of the SHG in continuum wave that results in the Manley-Rowe relation [26]

$$|e_2|^2 - \frac{1}{2} |e_1|^2 = \text{const}.$$

When SGH takes place in positive refractive index medium the minus in this expression is replaced to sign plus. It corresponds to the propagation of the both interacting waves in one direction.

## 4 Real form of the equations

It is suitable to introduce the real variables for the interacting waves

$$e_1 = a \exp(i\varphi_1), \quad e_2 = b \exp(i\varphi_2). \quad (10)$$

Substitution of these expressions into (8,9) leads to

$$\frac{\partial a}{\partial \zeta} + \frac{\sigma}{2} \left( 2 \frac{\partial a}{\partial \tau} \frac{\partial \varphi_1}{\partial \tau} + a \frac{\partial^2 \varphi_1}{\partial \tau^2} \right) = ab \sin \Phi, \quad (11)$$

$$\frac{\partial b}{\partial \zeta} + \delta \frac{\partial b}{\partial \tau} - \frac{\beta}{2} \left( 2 \frac{\partial b}{\partial \tau} \frac{\partial \varphi_2}{\partial \tau} + b \frac{\partial^2 \varphi_2}{\partial \tau^2} \right) = \frac{1}{2} a^2 \sin \Phi, \quad (12)$$

$$a \frac{\partial \varphi_1}{\partial \zeta} - \frac{\sigma}{2} \left( \frac{\partial^2 a}{\partial \tau^2} - a \frac{\partial \varphi_1}{\partial \tau} \frac{\partial \varphi_1}{\partial \tau} \right) = -ab \cos \Phi, \quad (13)$$

$$a \left( \frac{\partial \varphi_2}{\partial \zeta} + \delta \frac{\partial \varphi_2}{\partial \tau} \right) + \frac{\beta}{2} \left( \frac{\partial^2 b}{\partial \tau^2} + b \frac{\partial \varphi_2}{\partial \tau} \frac{\partial \varphi_2}{\partial \tau} \right) + \Delta b = \frac{1}{2} a^2 \cos \Phi, \quad (14)$$

where  $\Phi = \varphi_2 - 2\varphi_1$ . The case of continuum wave has been considered in [25, 26], where principal difference between spatial distribution of the pump and harmonic fields in the NRI medium with respect to positive refraction index medium was pointed.

## 5 Steady state solutions

Let us consider solutions without chirp, i.e., ones with  $\partial^2 \varphi_{1,2} / \partial \tau^2 = 0$ . Hence, we have  $\partial \varphi_{1,2} / \partial \tau = \Omega_{1,2}$ . The mismatch condition  $\Phi = 0$ , which is assumed additionally, leads to  $\Omega_2 = 2\Omega_1 = 2\Omega$ . Furthermore, we assume that  $\partial \varphi_{1,2} / \partial \zeta = \kappa_{1,2}$ , thus we suppose the phase functions evolve as

$$\varphi_1 = \kappa \zeta + \Omega \tau, \quad \varphi_2 = 2\kappa \zeta + 2\Omega \tau. \quad (15)$$

The amplitude equations (11) and (12) take the following form

$$\frac{\partial a}{\partial \zeta} + \sigma \Omega \frac{\partial a}{\partial \tau} = 0, \quad (16)$$

$$\frac{\partial b}{\partial \zeta} + (\delta - 2\beta \Omega) \frac{\partial b}{\partial \tau} = 0. \quad (17)$$

The steady state waves for both frequencies must propagate as single one. Thus, from (16, 17) we get the amplitudes as function of single variable  $y = \tau - \zeta/V$

$$a = a(\tau - \zeta/V), \quad b = b(\tau - \zeta/V)$$

with

$$V^{-1} = \sigma \Omega = \delta - 2\beta \Omega.$$

This expression defines the parameter "instant frequency"  $\Omega = \delta/(\sigma + 2\beta)$ .

The phase equations (13) and (14) take the following form

$$\frac{\partial^2 a}{\partial \tau^2} - (2\sigma\kappa + \Omega^2)a = 2\sigma ab, \quad (18)$$

$$\frac{\partial^2 b}{\partial \tau^2} + \frac{2}{\beta} (\Delta + 2\kappa + 2\delta\Omega - 2\beta\Omega^2) b = \frac{1}{\beta} a^2. \quad (19)$$

Let us suppose that  $b = fa$ . From (18) and (19) we get two equations for one function  $a = a(y)$  of one variable  $y = \tau - \zeta/V$  :

$$\frac{\partial^2 a}{\partial y^2} - (2\sigma\kappa + \Omega^2)a = 2\sigma fa^2, \quad (20)$$

$$\frac{\partial^2 a}{\partial y^2} + \frac{2}{\beta} (\Delta + 2\kappa + 2\delta\Omega - 2\beta\Omega^2) a = \frac{1}{\beta f} a^2 \quad (21)$$

These equations would define single function  $a$ , if the coefficients before equal orders of  $a$  are same, i.e.,  $f^2 = \sigma/2\beta > 0$  and

$$-(2/\beta) (\Delta + 2\kappa + 2\delta\Omega - 2\beta\Omega^2) = (2\sigma\kappa + \Omega^2). \quad (22)$$

It should be remarked that condition  $\sigma^2 = 1$  was used. This expression defines the parameter (effective wave number)  $\kappa$

$$\kappa = \frac{3\beta\Omega^2 - 4\delta\Omega - 2\Delta}{2\sigma(2\sigma + \beta)}.$$

From equation for function  $a(y)$  (20) we can obtain

$$(\partial a / \partial y)^2 = (2\sigma\kappa + \Omega^2)a^2 + (4\sigma f/3)a^3 + \text{const.} \quad (23)$$

If  $\sigma = +1$ , then dispersion parameter of second harmonic  $\beta$  must be positive too. Else, if  $\sigma = -1$ , then parameter  $\beta$  must be negative.

Let be  $\sigma = -1$ . In this case we have  $f = (1/2|\beta|)^{1/2}$ , and the equation (23) can be rewritten as

$$(\partial a / \partial y)^2 = (\Omega^2 - 2\kappa)a^2 - (4f/3)a^3 + \text{const.} \quad (24)$$

Let us introduce  $p = (\Omega^2 - 2\kappa)$  and  $w(y) = (4f/3p)a(y)$ . Consider the boundary condition as following:  $a \rightarrow a_0$ ,  $\partial a / \partial y \rightarrow 0$  at  $\tau \rightarrow \pm\infty$ . When

$a_0 = 0$ , we will get the solitary wave on zero background. The constant of integrating is defined now by following expression  $\text{const} = -p(w_0^2 - w_0^3)$ , where  $w_0 = (4f/3p)a_0$ . From equation (24) the equation for  $w(y)$  follows

$$(\partial w / \partial y)^2 = p(w - w_0) [\Delta^2 - (w - \gamma_0)^2], \quad (25)$$

where

$$\Delta^2 = (1 - w_0)(1 + 3w_0)/4, \quad \gamma_0 = (1 - w_0)/2.$$

If we introduce new variable  $u$  through the formula  $w = w_0 + u^{-2}$ , then the equation (25) transforms in to

$$(\partial u / \partial y)^2 = (p/4) [\Delta^2 u^4 - (1 - \gamma_1 u^2)^2], \quad (26)$$

where  $\gamma_1 = (1 - 3w_0)/2$ . Parameter  $\Delta^2$  is positive at  $0 \leq w_0 < 1$ .

## 5.1 Bright soliton solution

The solution of this kind corresponds with choosing  $w_0 = 0, \partial a / \partial y \rightarrow 0$  at  $\tau \rightarrow \pm\infty$ . In this case the parameters in (26) are  $\Delta^2 = 1/4$  and  $\gamma_1 = 1/2$ . The equation (26) transforms into following one

$$(\partial u / \partial y)^2 = (p/4)(u^2 - 1), \quad (27)$$

Put  $u = \cosh \vartheta$ , then from this equation we get  $(\partial \vartheta / \partial y)^2 = (p/4) > 0$ . Hence, solution of (27) is

$$u(y) = \cosh [p^{1/2}(y - y_0)/2], \quad (28)$$

where  $y_0$  is constant of integrating. Steady state solution of equations (11-14) in the case of  $\sigma = -1$  corresponding with *bright soliton* is

$$a(y) = \frac{(3p/4)\sqrt{2|\beta|}}{\cosh^2[\sqrt{p}(y - y_0)/2]}, \quad (29)$$

$$b(y) = \frac{(3p/4)}{\cosh^2[\sqrt{p}(y - y_0)/2]}. \quad (30)$$

If  $p = -|p| < 0$ , equation (27) has periodical solution

$$u(y) = \cos [\sqrt{|p|}(y - y_0)/2],$$



which leads to singular solution of the system (11- 14).

If  $\sigma = +1$  , then  $f = (1/2\beta)^{1/2}$  . The equation (23) takes the form

$$(\partial a/\partial y)^2 = (\Omega^2 + 2\kappa)a^2 + (4f/3)a^3 + \text{const.} \quad (31)$$

By introducing new function  $\tilde{w}(y) = -(4f/3p)a(y)$  from (31) the equation for  $\tilde{w}(y)$  follows

$$(\partial \tilde{w}/\partial y)^2 = \tilde{p}(\tilde{w} - \tilde{w}_0) \left[ \tilde{\Delta}^2 - (\tilde{w} - \tilde{\gamma}_0)^2 \right],$$

where  $\tilde{p} = (\Omega^2 + 2\kappa)$  ,  $\tilde{w}_0 = -(4f/3p)a_0$  , and

$$\tilde{\Delta}^2 = (1 - \tilde{w}_0)(1 + 3\tilde{w}_0)/4, \quad \tilde{\gamma}_0 = (1 - \tilde{w}_0)/2.$$

Hence, this case reduced to the one considered above.

## 5.2 Cnoidal wave solution

The solution of this kind corresponds with choosing  $0 < w_0 < 1$ . If  $0 < w_0 < 1/3$  then we have  $\Delta > \gamma_1 > 0$ . Equation (26) can be written as

$$(\partial u/\partial y)^2 = (p/4) [(\Delta + \gamma_1)u^2 - 1] [(\Delta - \gamma_1)u^2 + 1]. \quad (32)$$

Introduce the new variable  $\phi$  according to expression  $\cosh \phi = u\sqrt{\Delta + \gamma_1}$ . It leads to

$$(\partial \phi/\partial y)^2 = (p/4)(\Delta + \gamma_1) [1 + m^2 \cosh^2 \phi] \quad (33)$$

where  $m^2 = (\Delta - \gamma_1)(\Delta + \gamma_1)^{-1}$ . Substitution  $\phi = i\psi$  into this equation leads to

$$(\partial \psi/\partial y)^2 = -(p/4)(\Delta + \gamma_1)(1 + m^2) [1 - \tilde{m}^2 \sin^2 \psi].$$

It is equation for the Jacobi elliptic functions. Hence, we can write the solution by following formula

$$\sin \psi = \text{sn}(iz, \tilde{m}), \quad (34)$$

where  $z = \sqrt{p(\Delta + \gamma_1)(1 + m^2)}(y - y_0)/2$ ,  $\tilde{m}^2 = m^2(1 + m^2)^{-1}$ . By using the properties of the trigonometric functions and Jacobi elliptic functions, in

particular,  $\text{sn}(iz, k) = \text{isc}(z, k'), k'^2 + k^2 = 1$ , and  $\text{sn}^2(z, k) + \text{cn}^2(z, k) = 1$ , we can write

$$\sinh \phi = -\frac{\text{sn}(z, \tilde{m}')}{\text{cn}(z, \tilde{m}')}.$$

Now, we have

$$u^2(y) = \frac{1}{(\Delta + \gamma_1)\text{cn}^2(z, \tilde{m}')}.$$
 (35)

Thus, the solution of the equation (25) represents by formula

$$w(y) = w_0 + (\Delta + \gamma_1)\text{cn}^2(z, \tilde{m}').$$
 (36)

By using definitions of the parameter  $\tilde{m}$  and  $m^2$  one can find that

$$\tilde{m}' = \left[ \frac{1}{2} \left( 1 + \frac{\gamma_1}{\Delta} \right) \right]^{1/2}, \quad z = \left( \frac{2p\Delta}{\Delta + \gamma_1} \right)^{1/2} (y - y_0).$$

At interval  $0 < w_0 < 1/3$  module of Jacobi elliptic functions is not equal to unit, hence only cnoidal wave exists at these values of the pedestal amplitude  $w_0$ .

The solution of the equation (26) under condition that  $1/3 < w_0 < 1$  can be found in a similar way. If  $1/3 < w_0 < 2/3$  this solution represents by formula

$$w(y) = w_0 + (\Delta - |\gamma_1|)\text{cn}^2(z_1, m_1),$$
 (37)

where

$$m_1 = \left[ \frac{1}{2} \left( 1 - \frac{|\gamma_1|}{\Delta} \right) \right]^{1/2}, \quad z_1 = \left( \frac{p\Delta}{2} \right)^{1/2} (y - y_0).$$

If  $2/3 < w_0 < 1$  we can find

$$w(y) = w_0 + (\Delta + |\gamma_1|) \frac{\text{cn}^2(z_2, m_2)}{\text{sn}^2(z_2, m_2)},$$
 (38)

where

$$m_2 = \left[ \frac{2\Delta}{\Delta + |\gamma_1|} \right]^{1/2}, \quad z_2 = \frac{1}{2} [p|(\Delta + |\gamma_1|)]^{1/2} (y - y_0).$$

This solution describes the cnoidal wave, however amplitude of this wave periodically attains the infinite value. Hence, this solution is not acceptable for considered there physical situation. If  $w_0 > 1$  equation (26) has only singular (unlimited) solutions.

### 5.3 Dark soliton solution

The quadratic dark soliton has been found in [33, 34]. The similar solution can be found in the NRI medium. Let consider the equation (25). Substitution in  $w = w_0 + q$  results in the following equation

$$(\partial q / \partial y)^2 = pq [\Delta^2 - (q - \gamma_1)^2]. \quad (39)$$

By following [34] let us find the solution in the form  $q = A / \cosh \alpha y$ . If substitute that in (39), and equating the coefficients of various powers of  $\cosh \alpha y$  in this equation to zero, we obtain the following system of relations

$$p(\Delta^2 - \gamma_1^2) = 0, \quad 2p\gamma_1 A = 4\alpha^2 A, \quad A^2 = 4\alpha A. \quad (40)$$

From these relations it follows that  $\alpha^2 = p\gamma_1/2$ ,  $A = 4\alpha$ , and  $\Delta^2 - \gamma_1^2 = 0$ . By using the definition of  $\Delta^2$  and  $\gamma_1$  we can find that equation  $\Delta^2 = \gamma_1^2$  will be held at  $w_0 = 0$  and  $w_0 = 2/3$ . In the past case  $\gamma_1 = -1/2$ . From relation  $\alpha^2 = p\gamma_1/2$  it follows that  $p$  must be negative, and  $\alpha = \pm(|p|/4)^{1/2}$ . Hence, we have  $A = \pm 2(|p|)^{1/2}$ . Thus,

$$q = \frac{\pm 2\sqrt{|p|}}{\cosh^2[\sqrt{|p|}y/2]}.$$

The solution of equations (11- 14) in this case is

$$a(y) = (3|p|/4)\sqrt{2|\beta|} \left( 2/3 \pm 2\sqrt{|p|} \operatorname{sech}^2[\sqrt{|p|}y/2] \right) \quad (41)$$

$$b(y) = (3|p|/4) \left( 2/3 \pm 2\sqrt{|p|} \operatorname{sech}^2[\sqrt{|p|}y/2] \right) \quad (42)$$

These expressions describe the quadratic dark soliton propagating in nonlinear medium with negative refractive index.

### 5.4 Two-hump soliton

Up to this point the solutions of equations (18) and (19) have been found under constraint  $b = fa$ . It is known [35] that for positive refractive index medium there are solutions where this constraint is absent. These solutions were named by *simultons* in [28]. We can show that the similar solutions

exist in NRIM too. Follow [35] we assume that solution of the (18) and (19) has the form

$$a(y) = A \tanh \alpha y \sec h \alpha y, \quad (43)$$

$$b(y) = B \sec h^2 \alpha y, \quad (44)$$

where  $A$ ,  $B$  and  $\alpha$  are unknown parameters. Substitution of this ansatz in (18) and (19), and grouping the terms with the equal powers of  $\cosh \alpha y$  and  $\tanh \alpha y \sec h \alpha y$  in these equations to zero, provides the following system of algebraic relations

$$\alpha^2 A - p_1 A = 0, \quad (45)$$

$$2\sigma AB + 6\alpha^2 A = 0, \quad (46)$$

$$A^2/\beta - 6\alpha^2 A = 0, \quad (47)$$

$$-A^2/\beta + 4\alpha^2 B = p_2, \quad (48)$$

where  $(2\sigma\kappa + \Omega^2) = p_1$  and  $(2/\beta)(\Delta + 2\kappa + 2\delta\Omega - 2\beta\Omega^2) = p_2$ . It leads to

$$\begin{aligned} \alpha^2 &= p_1 > 0, & A^2 &= 6\beta\alpha^2 B, \\ B &= p_2/2\alpha^2, & B &= -3\alpha^2\sigma. \end{aligned}$$

From these expression we obtain  $A^2 = -18\beta\alpha^4\sigma$ . It means that if  $\sigma = -1$  then  $\beta > 0$ . Else, we can choose  $\sigma = 1$ , then  $\beta < 0$  and  $p_2 = -|p_2| < 0$ .

Let be  $\sigma = -1$ . In this case we can found

$$\alpha^2 = p_1, \quad B = 3p_1, \quad A = 3\sqrt{2\beta}p_1, \quad (49)$$

and relation

$$p_2 = 6p_1^2. \quad (50)$$

From (50) it follows

$$(\Delta + 2\kappa + 2\delta\Omega - 2\beta\Omega^2) = 3\beta(2\kappa - \Omega^2)^2. \quad (51)$$

The parameter  $\Omega$  has been defined as  $\Omega = \delta/(2\beta - 1)$ , hence relation (51) defines parameter through phase mismatch  $\Delta$ . From (51) it follows

$$2\kappa^{(\pm)} = \frac{1 + 6\beta\Omega^2 \pm \sqrt{1 + 3\beta(\Delta + 2(3\beta - 1)\Omega^2 + 9\beta\Omega^4)}}{6\beta}. \quad (52)$$

Thus, the two-hump soliton has the following form

$$a(y) = 3p_1\sqrt{2\beta}\tanh(\sqrt{p_1}y)\operatorname{sech}(\sqrt{p_1}y), \quad (53)$$

$$b(y) = 3p_1\operatorname{sech}^2(\sqrt{p_1}y). \quad (54)$$

For example, let us assume that  $\beta = 1$ , then  $\Delta = -\delta^2$ ,  $2\kappa^{(\pm)} = \delta^2 \pm 1/3$  and  $p_1^{(\pm)} = \pm 1/3$ . Only  $p_1^{(+)} = 1/3$  is acceptable, hence, we get the two-hump solution of the system (18) and (19)

$$a(y) = \sqrt{2}\tanh(y/\sqrt{3})\operatorname{sech}(y/\sqrt{3}), \quad (55)$$

$$b(y) = \operatorname{sech}^2(y/\sqrt{3}). \quad (56)$$

For the case of the positive refractive index medium there is the solution describing the perimetrically connected waves of the pump and second harmonic. Analogue of that exists in the NRI medium too.

## 6 Conclusion

We considered the steady state propagation of the coupled pair of waves, i.e., the pump wave at the frequency  $\omega_0$  and wave at the second harmonic frequency, in the quadratic nonlinear NRIM. In this case the group velocities of pump and harmonic pulses are directed in opposite directions [23, 24, 25, 26]. We are found that under some conditions the self trapping of the interacting wave packets takes place. Two frequency wave packet (simulton) is generated from initial pulses alike to quadratic soliton of positive refractive medium (PRIM).

To do comparison of the solitons in quadratic NRIM with solitons in PRIM it is suitable to represent the appropriate expressions for second case. The system of equations describing the quadratic solitons propagation in PRIM medium can be written as

$$i\frac{\partial}{\partial\zeta}e_1 - \frac{\sigma}{2}\frac{\partial^2}{\partial\tau^2}e_1 + e_1^*e_2 = 0 \quad (57)$$

$$i\left(\frac{\partial}{\partial\zeta} + \delta\frac{\partial}{\partial\tau}\right)e_2 - \frac{\beta}{2}\frac{\partial^2}{\partial\tau^2}e_2 - \Delta e_2 + \frac{1}{2}e_1^2 = 0 \quad (58)$$

where  $\delta = Lt_p^{-1}(v_1^{-1} - v_2^{-1}) = t_p|D_1|^{-1}(v_1^{-1} - v_2^{-1})$ . In NRIM parameter  $\delta$  is always positive. Hence, there is no solutions corresponding with solitons of PRIM which have  $\delta = 0$

Velocities of the bright solitons in NRIM and PRIM are differed. For NRIM all steady state pulses move with velocity

$$\frac{1}{V_s} = \frac{1}{\sigma + 2\beta} \left( \frac{\sigma}{v_2} - \frac{2\beta}{v_1} \right) = \frac{1}{D_1 + 2D_2} \left( \frac{D_1}{v_2} - \frac{2D_2}{v_1} \right). \quad (59)$$

Whereas in PRI medium bright soliton has the velocity

$$\frac{1}{V_s} = \frac{1}{2\beta - \sigma} \left( \frac{2\beta}{v_1} - \frac{\sigma}{v_2} \right) = \frac{1}{2D_3 - D_1} \left( \frac{2D_2}{v_1} - \frac{D_1}{v_2} \right). \quad (60)$$

In the plane  $(\delta, \Delta)$  the existence regions of the steady state wave are differed also. For example, if one put dispersion parameters as follows  $\sigma = \beta = -1$  in the case of NRI medium from (29-30) the expressions for envelopes of bright soliton result in

$$a(y) = \frac{\sqrt{2}(3\Delta - \delta^2)}{6 \cosh^2[\sqrt{(3\Delta - \delta^2)/2}(y - y_0)/3]}, \quad (61)$$

$$b(y) = \frac{(3\Delta - \delta^2)}{6 \cosh^2[\sqrt{(3\Delta - \delta^2)/2}(y - y_0)/3]}. \quad (62)$$

This solution exists under condition  $\Delta > \delta^2/3$ . The solution of the equation (57) and (58) yields expression for quadratic soliton in PRIM

$$a(y) = \frac{3\sqrt{2}(\Delta + \delta^2)}{2 \cosh^2[\sqrt{(\Delta + \delta^2)/2}(y - y_0)]}, \quad (63)$$

$$b(y) = \frac{3(\Delta + \delta^2)}{2 \cosh^2[\sqrt{(\Delta + \delta^2)/2}(y - y_0)]}. \quad (64)$$

In this case the solution exists under condition  $\Delta + \delta^2 > 0$ . Thus the NRIM solitons and PRIM solitons exist in different regions of parameters plane  $(\delta, \Delta)$ . Stability conditions we plane to consider in future investigation. It is possible that the stability regions will be different too.

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